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DIFFERENT REPRESENTATION OF GROWTH INDICATORS OF ENTIRE FUNCTIONS UNDER THE FLAVOUR OF P-ADIC ANALYSIS

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Abstract
Let K be a complete ultametric algebraically closed field and A(K) be the K-algebra of entire functions on K. For $f, g \in \mathrm{A(K)}$, in this paper, we wish to establish another representation of order and lower order of $f \in \mathrm{A(K)}$. Also we establish the integral representation of generalized (p, q)-th relative type and generalized (p, q)-th relative weak type of entire function f with respect to another entire function g, where $f, g \in \mathrm{A(K)}$. We also establish their equivalence relation under some certain condition.

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1 Introduction
Let us consider K to be an algebraically closed field of characteristic 0, complete with respect to a p-adic absolute value $|.|$ (example $\mathbb{C}_p$). For any $\alpha \in \mathrm{K}$ and $r \in]0, +\infty[$, the closed disc $\{x \in K: |x - a| \leq R\}$ and the open disc $\{x \in K: |x - a| < R\}$ are denoted by $d(\alpha, R)$ and $d(\alpha, R^-)$ respectively. Also $C(\alpha, R)$ denotes the circle $\{x \in K: |x - a| = R\}$. Moreover, A(K) represents the K-algebra of analytic functions in K, i.e., the set of power series with an infinite radius of convergence. During the last several years, the idea of p-adic analysis have been studied from different aspects and we get many important results from {cf. [2], [3], [5], [6], [8] and [9]}.
Let $f \in \mathrm{A(K)}$ and $r > 0$, then we denote by $|f|(r)$ the number $\sup\{|f(x)|: |x| = r\}$, where $|.|(r)$ is a multiplicative norm on $A(K)$. Moreover, if f is not constant the $|f|(r)$is a strictly increasing function of r and tends to $+\infty$ with r. So there exists its inverse function $\widehat{|f|}: (|f(0)|, \infty) \rightarrow (0, \infty)$ with $\lim_{s\rightarrow\infty} \widehat{|f|}(s) = \infty$.
Therefore, for any two entire functions $f \in \mathrm{A(K)}$ and $g \in \mathrm{A(K)}$ the ratio $\frac{|f|(r)}{|g|(r)}$ as $r \rightarrow \infty$ is called the growth of f with respect to g in terms of their multiplicative norm.
For $x \in [0, \infty)$ and $k \in \mathbb{N}$, Biswas {cf. [2]} defined
$\log^{[k]} x = \log(\log^{[k-1]} x)$ and $exp^{[k]} x = exp\left(exp^{[k-1]} x\right)$,
where $\mathbb{N}$ be the set of all positive integers. We also denote $\log^{[0]} x = x$ and $exp^{[0]} x = x$. Throughout the paper, log denotes the Neperian logarithm. Taking this into account the order (resp. lower order) of an entire function $f \in \mathrm{A(K)}$) is given by {cf. [1], [2] and [7]}.
$\rho(f) = \lim_{r\rightarrow\infty} \sup \frac{\log^{[2]} |f|(r)}{\log r}$ and $\lambda(f) = \lim_{r\rightarrow\infty} \inf \frac{\log^{[2]} |f|(r)}{\log r}$.

Boussaf et al.{cf. [1]} also introduce the definition of type (resp. lower type) of an entire function $f \in$ A(K), which is also another type of growth indicator used for comparing the relative growth of two entire function defined in $A(K)$. having same non zero finite order in the following way,
$\sigma(f) = \lim_{r\to+\infty} \sup \frac{\log|f|(r)}{r^{\rho(f)}}$ and $\bar{\sigma}(f) = \lim_{r\to+\infty} \inf \frac{\log|f|(r)}{r^{\rho(f)}}$, where $0 < \rho(f) < \infty$.
Analogously for $0 < \lambda(f) < \infty$, one may give the definition of weak type $\tau(f)$ and growth indicator $\bar{\tau}(f)$ as
$\tau(f) = \lim_{r\to+\infty} \inf \frac{\log|f|(r)}{r^{\lambda(f)}}$ and $\bar{\tau}(f) = \lim_{r\to+\infty} \sup \frac{\log|f|(r)}{r^{\lambda(f)}}$.

Definition 1.1 ({cf. [4]}): Let $f, g \in$ A(K). The relative (p,q)-th order and (p,q)-th lower order of entire function f with respect to another entire function g are defined as
$\rho_g^{(p,q)}(f) = \lim_{r\to\infty} \sup \frac{\log^{[p]}|\widehat{g}|(|f|(r))}{\log^{[q]} r}$ and $\lambda_g^{(p,q)}(f) = \lim_{r\to\infty} \inf \frac{\log^{[p]}|\widehat{g}|(|f|(r))}{\log^{[q]} r}$
where p, q are two positive integers. Further for any $f \in$ A(K) for which (p,q)-th relative order and (p,q)-th relative lower order with respect to $g \in$ A(K) are the same is called a function of regular relative (p,q)-th growth with respect to g otherwise f is said to be irregular relative (p,q)-th growth with respect to g.

Definition 1.2 ({cf. [4]}): Let $f, g \in$ A(K). The (p,q)-th relative type of f with respect to g having finite positive (p,q)-th relative order $\sigma_g^{(p,q)}(f)$ is defined as
$\sigma_g^{(p,q)}(f) = \lim_{r\to+\infty} \sup \frac{\log^{[p-1]}|\widehat{g}|(|f|(r))}{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}}$, where p, q are any two positive integers.

Definition 1.3: Let $f, g \in$ A(K) be any two entire functions having finite positive (p,q)-th relative generalized order $\rho_g^{(p,q)}(f)$, $\left(0 < \rho_g^{(p,q)}(f) < \infty\right)$ where p, q are any two positive integers. Then (p,q)-th relative generalized type $\sigma_g^{(p,q)}(f)$ of entire function f with respect to the entire function g is defined as: the integral
$$\int_{r_0}^{\infty} \frac{\log^{[p-2]}|\widehat{g}|\big(|f|(r)\big)}{\left[\exp\left\{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{k+1}} dr, \quad (r_0 > 0)$$
is convergent for $k > \sigma_g^{(p,q)}(f)$ and divergent for $k < \sigma_g^{(p,q)}(f)$.

Definition 1.4 ({cf. [4]}): Let $f, g \in$ A(K). The (p,q)-th relative generalized weak type $\sigma_g^{(p,q)}(f)$, of entire function f with respect to the entire function g having finite positive (p, q)-th relative lower order $\lambda_g^{(p,q)}(f)$ is defined as
$\tau_g^{(p,q)}(f) = \lim_{r\to+\infty} \inf \frac{\log^{[p-1]}|\widehat{g}|(|f|(r))}{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}}$, where p, q are any two positive integers.

Definition 1.5: Let $f, g \in$ A(K) be any two entire functions having finite positive (p,q)-th relative generalized lower order $\lambda_g^{(p,q)}(f)$, $\left(0 < \lambda_g^{(p,q)}(f) < \infty\right)$ where p, q are any two positive integers. Then (p,q)-th relative generalized weak type $\tau_g^{(p,q)}(f)$ of entire function f with respect to the entire function g is defined as: the integral
$$\int_{r_0}^{\infty} \frac{\log^{[p-2]}|\widehat{g}|\big(|f|(r)\big)}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{k+1}} dr, \quad (r_0 > 0)$$
is convergent for $k > \tau_g^{(p,q)}(f)$ and divergent for $k < \tau_g^{(p,q)}(f)$.

2 Material and Methods

Definition 2.1 ({cf. [4]}): Let $f, g \in$ A(K). The (p,q)-th relative type of f with respect to g having finite positive (p,q)-th relative order $\bar{\sigma}_g^{(p,q)}(f)$ is defined as

$\bar{\sigma}_g^{(p,q)}(f) = \lim_{r\to+\infty} \inf \frac{\log^{[p-1]}|\widehat{g}|(|f|(r))}{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}}$, where p, q are any two positive integers.

Definition 2.2: Let $f, g \in$ A(K) be any two entire functions having finite positive (p,q)-th relative generalized order $\rho_g^{(p,q)}(f)$, $\left(0 < \rho_g^{(p,q)}(f) < \infty\right)$ where p, q are any two positive integers. Then (p,q)-th relative generalized lower type $\bar{\sigma}_g^{(p,q)}(f)$ of entire function f with respect to the entire function g is defined as: the integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]}|\widehat{g}|\big(|f|(r)\big)}{\left[\exp\left\{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{k+1}} dr, \quad (r_0 > 0)$$

is convergent for $k > \bar{\sigma}_g^{(p,q)}(f)$ and divergent for $k < \bar{\sigma}_g^{(p,q)}(f)$.

Definition 2.3 ({cf. [4]}): Let $f, g \in$ A(K). The growth indicator $\bar{\tau}_g^{(p,q)}(f)$ of entire function f with respect to the entire function g having finite positive (p,q)-th relative lower order $\lambda_g^{(p,q)}(f)$ is defined as

$\bar{\tau}_g^{(p,q)}(f) = \lim_{r\to+\infty} \sup \frac{\log^{[p-1]}|\widehat{g}|(|f|(r))}{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}}$, where p, q are any two positive integers.

Definition 2.4: Let $f, g \in$ A(K) be any two entire functions having finite positive (p,q)-th relative generalized order $\lambda_g^{(p,q)}(f)$, $\left(0 < \lambda_g^{(p,q)}(f) < \infty\right)$ where p, q are any two positive integers. Then the growth indicator $\bar{\tau}_g^{(p,q)}(f)$ of entire function f with respect to the entire function g is defined as: the integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]}|\widehat{g}|\big(|f|(r)\big)}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{k+1}} dr, \quad (r_0 > 0)$$

is convergent for $k > \bar{\tau}_g^{(p,q)}(f)$ and divergent for $k < \bar{\tau}_g^{(p,q)}(f)$.

Lemma 2.1: Let $f, g \in$ A(K) be any two entire functions and let the integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]}|\widehat{g}|\big(|f|(r)\big)}{[\exp\{(\log^{[q-1]} r)^{A}\}]^{k+1}} dr, \quad (r_0 > 0)$$

is convergent for $0 < A < \infty$. Then

$$\lim_{r\to\infty} \frac{\log^{[p-2]}|\widehat{g}|\big(|f|(r)\big)}{[\exp\{(\log^{[q-1]} r)^{A}\}]^{k}} = 0.$$

Proof: Since the integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]}|\widehat{g}|\big(|f|(r)\big)}{[\exp\{(\log^{[q-1]} r)^{A}\}]^{k+1}} dr, \quad (r_0 > 0)$$

is convergent then

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]}|\widehat{g}|\big(|f|(r)\big)}{[\exp\{(\log^{[q-1]} r)^{A}\}]^{k+1}} dr < \epsilon \; if \quad r_0 > R(\epsilon).$$

$$i.e., \qquad \int_{r_0}^{\exp\left\{(\log^{[q-1]} r_0)^{A}\right\}+r_0} \frac{\log^{[p-2]}|\widehat{g}|\big(|f|(r)\big)}{[\exp\{(\log^{[q-1]} r)^{A}\}]^{k+1}} dr < \epsilon.$$

Since $\log^{[p-2]}\widehat{[g]}\left(|f|(r)\right)$ increases with r, so

$$\int_{r_0}^{\exp\left\{\left(\log^{[q-1]} r_0\right)^A\right\}+r_0} \frac{\log^{[p-2]}\widehat{[g]}\left(|f|(r)\right)}{[\exp\{(\log^{[q-1]} r)^A\}]^{k+1}} dr \geq \frac{\log^{[p-2]}\widehat{[g]}\left(|f|(r_0)\right)}{[\exp\{(\log^{[q-1]} r_0)^A\}]^{k+1}} . \exp\left\{\left(\log^{[q-1]} r_0\right)^A\right\}$$

i.e., for all large values of r,

$$\int_{r_0}^{\exp\left\{\left(\log^{[q-1]} r_0\right)^A\right\}+r_0} \frac{\log^{[p-2]}\widehat{[g]}\left(|f|(r)\right)}{[\exp\{(\log^{[q-1]} r)^A\}]^{k+1}} dr \geq \frac{\log^{[p-2]}\widehat{[g]}\left(|f|(r_0)\right)}{[\exp\{(\log^{[q-1]} r_0)^A\}]^{k}}$$

so that

$$\frac{\log^{[p-2]}\widehat{[g]}\left(|f|(r_0)\right)}{[\exp\{(\log^{[q-1]} r_0)^A\}]^{k}} < \epsilon \; if \quad r_0 > R(\epsilon).$$

$$i.e., \qquad \lim_{r\to\infty} \frac{\log^{[p-2]}\widehat{[g]}\left(|f|(r)\right)}{[\exp\{(\log^{[q-1]} r)^A\}]^{k}} = 0.$$

This proves the lemma.

Now a question may arise about the equivalence of definitions of (p,q)-th relative generalized type and (p,q)-th relative generalized weak type with their integral representations. In this paper, we would like to establish such equivalence of Definition 1.2 and Definition 1.3, and Definition 2.1 and Definition 2.2. Here we also investigate some growth properties related to (p,q)-th relative generalized type and (p,q)-th relative generalized weak type of entire function with respect to another entire function.

3 Results and Discussion

In this section we establish the main results of the paper.

Theorem 3.1: If $f \in$ A(K) then $\rho(f) = \sup\left\{s \in]0,\infty[\,\middle|\, \lim_{r\to\infty} \sup \frac{\log|f|(r)}{r^s} > 0\right\}$.

Proof: Let $P = \sup\left\{s \in]0,\infty[\,\middle|\, \lim_{r\to\infty} \sup \frac{\log|f|(r)}{r^s} > 0\right\}$.

Let us suppose that for some $s > 0$, we have

$$\lim_{r\to\infty} \sup \frac{\log|f|(r)}{r^s} = b > 0$$

From Definition of supremum, we have for arbitrary ϵ and for all large values of r,

$$\frac{\log|f|(r)}{r^s} \leq b + \epsilon$$

$$i.e., \qquad \frac{\log(\log|f|(r))}{\log r} \leq s + \frac{\log(b+\epsilon)}{\log r}. \tag{3.1}$$

Again, for a sequence of values of r tending to ∞

$$\frac{\log|f|(r)}{r^s} \geq b - \epsilon$$

$$i.e., \qquad \frac{\log(\log|f|(r))}{\log r} \geq s + \frac{\log(b-\epsilon)}{\log r}. \tag{3.2}$$

Combining Equation (3.1) and Equation (3.2) we get

$$s + \frac{\log(b-\epsilon)}{\log r} \leq \frac{\log(\log|f|(r))}{\log r} \leq s + \frac{\log(b+\epsilon)}{\log r}.$$

Since $\epsilon > 0$ is arbitrary we get that

$$\lim_{r\to\infty} \sup \frac{\log(\log|f|(r))}{\log r} = P$$

$$i.e., \qquad \rho(f) = P.$$

This proves the theorem.

Theorem 3.2: If $f \in \mathrm{A(K)}$ then $\lambda(f) = \inf\left\{s \in]0,\infty[\,\middle|\, \liminf_{r\to\infty} \frac{\log|f|(r)}{r^s} > 0\right\}$.

Proof: Let $Q = \inf\left\{s \in]0,\infty[\,\middle|\, \liminf_{r\to\infty} \frac{\log|f|(r)}{r^s} > 0\right\}$.

Let us suppose that for some $s > 0$, we have

$$\liminf_{r\to\infty} \frac{\log|f|(r)}{r^s} = d > 0.$$

From Definition of infimum, we have for arbitrary ϵ and for all large values of r,

$$\frac{\log|f|(r)}{r^s} \geq d - \epsilon$$

$$i.e., \quad \frac{\log(\log|f|(r))}{\log r} \geq s + \frac{\log(d-\epsilon)}{\log r}. \tag{3.3}$$

Again, for a sequence of values of r tending to ∞

$$\frac{\log|f|(r)}{r^s} \leq d + \epsilon$$

$$i.e., \quad \frac{\log(\log|f|(r))}{\log r} \leq s + \frac{\log(d+\epsilon)}{\log r}. \tag{3.4}$$

Combining Equation (3.3) and Equation (3.4) we obtain that

$$s + \frac{\log(d-\epsilon)}{\log r} \leq \frac{\log(\log|f|(r))}{\log r} \leq s + \frac{\log(d+\epsilon)}{\log r}.$$

Since $\epsilon > 0$ is arbitrary we get that

$$\liminf_{r\to\infty} \frac{\log(\log|f|(r))}{\log r} = Q$$

$$i.e., \quad \lambda(f) = Q.$$

which proves the theorem.

Theorem 3.3: Let $f, g \in \mathrm{A(K)}$ be any two entire functions having finite positive (p,q)-th relative generalized order $\rho_g^{(p,q)}(f)$, $\left(0 < \rho_g^{(p,q)}(f) < \infty\right)$ and (p,q)-th relative generalized type $\sigma_g^{(p,q)}(f)$ where p, q are any two positive integers. Then Definition 1.2 and Definition 1.3 are equivalent.

Proof: Let $f, g \in \mathrm{A(K)}$ be any two entire functions such that $\rho_g^{(p,q)}(f)$, $\left(0 < \rho_g^{(p,q)}(f) < \infty\right)$ exists, where p, q are any two positive integers.

Case I: Let $\sigma_g^{(p,q)}(f) = \infty$.

Definition 1.2 $\Longrightarrow$ Definition 1.3.

As $\sigma_g^{(p,q)}(f) = \infty$, from Definition (1.2) we have for an arbitrary $G > 0$ and a sequence of values of r tending to infinity,

$$\log^{[p-1]} \widehat{|g|}\left(|f|(r)\right) > G.\left(\log^{[q-1]} r\right)^{\rho_g^{(p,q)}(f)}$$

$$i.e., \quad \log^{[p-1]} \widehat{|g|}\left(|f|(r)\right) > \left[\exp\left\{\left(\log^{[q-1]} r\right)^{\rho_g^{(p,q)}(f)}\right\}\right]^G. \tag{3.5}$$

If possible, let the integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} \widehat{|g|}\left(|f|(r)\right)}{\left[\exp\left\{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{G+1}} dr, \quad (r_0 > 0)$$

be converge. Then by Lemma 2.1, we get that

$$\lim_{r\to\infty}\sup\frac{\log^{[p-2]}\widehat{|g|}\big(|f|(r)\big)}{\left[\exp\left\{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{G}}=0.$$

So, for all sufficiently large values of r,

$$\log^{[p-2]}\widehat{|g|}\big(|f|(r)\big)<\left[\exp\left\{\left(\log^{[q-1]} r\right)^{\rho_g^{(p,q)}(f)}\right\}\right]^{G}. \qquad (3.6)$$

Therefore, by Equation (3.5) and Equation (3.6) we arrive at a contradiction.
Hence

$$\int_{r_0}^{\infty}\frac{\log^{[p-2]}\widehat{|g|}\big(|f|(r)\big)}{\left[\exp\left\{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{G+1}}dr,\ \ (r_0>0)$$

is divergent where $G>0$ is finite, which is Definition 1.3.
Now we show Definition 1.3 ⟹ Definition 1.2.
Let G be any positive number. Since $\sigma_g^{(p,q)}(f)=\infty$, from Definition 1.3 the divergence of the integral

$$\int_{r_0}^{\infty}\frac{\log^{[p-2]}\widehat{|g|}\big(|f|(r)\big)}{\left[\exp\left\{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{G+1}}dr,\ \ (r_0>0)$$

gives an arbitrary positive ϵ and for a sequence of values of r tending to infinity

$$\log^{[p-2]}\widehat{|g|}\big(|f|(r)\big)>\left[\exp\left\{\left(\log^{[q-1]}r\right)^{\rho_g^{(p,q)}(f)}\right\}\right]^{G-\epsilon}$$

$$i.e., \qquad \log^{[p-1]}\widehat{|g|}\big(|f|(r)\big)>(G-\epsilon)\left(\log^{[q-1]}r\right)^{\rho_g^{(p,q)}(f)}$$

$$i.e., \qquad \frac{\log^{[p-1]}\widehat{|g|}\big(|f|(r)\big)}{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}}>G-\epsilon.$$

Since $G>0$ is arbitrary, it follows that

$$\lim_{r\to\infty}\sup\frac{\log^{[p-1]}\widehat{|g|}\big(|f|(r)\big)}{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}}=\infty$$

$$i.e., \qquad \sigma_g^{(p,q)}(f)=\infty.$$

Thus Definition 1.2 follows.
Case II: Let $0\le\sigma_g^{(p,q)}(f)<\infty$.
First, we show that Definition 1.2 ⟹ Definition 1.3.
Sub case (A): Let $0<\sigma_g^{(p,q)}(f)<\infty$.
Let $f,g\in \mathrm{A(K)}$ be any two entire functions such that $0<\sigma_g^{(p,q)}(f)<\infty$ exists for positive integers p, q. Then according to Definition 1.2 for any arbitrary positive ϵ and for large value of r we obtain that

$$\log^{[p-1]}\widehat{|g|}\big(|f|(r)\big)<\left(\sigma_g^{(p,q)}(f)+\epsilon\right)\left\{\log^{[q-1]}r\right\}^{\rho_g^{(p,q)}(f)}$$

$$i.e., \qquad \frac{\log^{[p-2]}\widehat{|g|}\big(|f|(r)\big)}{\left[\exp\left\{(\log^{[q-1]}r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{k}}<\frac{1}{\left[\exp\left\{(\log^{[q-1]}r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{k-\left(\sigma_g^{(p,q)}(f)+\epsilon\right)}}$$

Therefore,

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} \widehat{|g|}\big(|f|(r)\big)}{\left[\exp\left\{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{k+1}} dr, \ \ (r_0 > 0)$$

is convergent for $k > \sigma_g^{(p,q)}(f)$.

Again, by Definition 1.2 we obtained for a sequence of values of r tending to infinity that

$$\log^{[p-1]} \widehat{|g|}\big(|f|(r)\big) > \left(\sigma_g^{(p,q)}(f) - \epsilon\right)\left\{\log^{[q-1]} r\right\}^{\rho_g^{(p,q)}(f)}$$

$$i.e., \quad \log^{[p-2]} \widehat{|g|}\big(|f|(r)\big) > \left[\exp\left\{\left(\log^{[q-1]} r\right)^{\rho_g^{(p,q)}(f)}\right\}\right]^{\left(\sigma_g^{(p,q)}(f) - \epsilon\right)}, \tag{3.7}$$

so for $k < \sigma_g^{(p,q)}(f)$, we get from Equation (3.7) that

$$\frac{\log^{[p-2]} \widehat{|g|}\big(|f|(r)\big)}{\left[\exp\left\{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{k}} > \frac{1}{\left[\exp\left\{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{k - \left(\sigma_g^{(p,q)}(f) - \epsilon\right)}}$$

$$i.e., \quad \int_{r_0}^{\infty} \frac{\log^{[p-2]} \widehat{|g|}\big(|f|(r)\big)}{\left[\exp\left\{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{k+1}} dr, \ \ (r_0 > 0)$$

is divergent for $k < \sigma_g^{(p,q)}(f)$.

Hence

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} \widehat{|g|}\big(|f|(r)\big)}{\left[\exp\left\{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{k+1}} dr, \ \ (r_0 > 0)$$

is convergent for $k > \sigma_g^{(p,q)}(f)$ and divergent for $k < \sigma_g^{(p,q)}(f)$.

Sub case (B): Let $\sigma_g^{(p,q)}(f) = 0$.

When $\sigma_g^{(p,q)}(f) = 0$ for positive integer p, q Definition (1.2) gives for all sufficiently large values of r that

$$\frac{\log^{[p-2]} \widehat{|g|}\big(|f|(r)\big)}{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}} < \epsilon.$$

Then similar as before we get that

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} \widehat{|g|}\big(|f|(r)\big)}{\left[\exp\left\{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{k+1}} dr, \ \ (r_0 > 0)$$

is convergent for $k > 0$ and divergent for $k < 0$.

Thus, combining subcase (A) and subcase (B) Definition 1.3 follows.

Now we show Definition 1.3 $\Longrightarrow$ Definition 1.2.

From Definition 1.3 and arbitrary positive ϵ, the integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} \widehat{|g|}\big(|f|(r)\big)}{\left[\exp\left\{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{\sigma_g^{(p,q)}(f) + \epsilon + 1}} dr, \ \ (r_0 > 0)$$

is convergent. Then by Lemma 2.1 we get that

$$\frac{\log^{[p-2]}\left|\widehat{g}\right|\left(|f|(r)\right)}{\left[\exp\left\{(\log^{[q-1]}r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{\sigma_g^{(p,q)}(f)+\epsilon}}=0.$$

So, we obtain for all sufficiently large values of r that

$$\frac{\log^{[p-2]}\left|\widehat{g}\right|\left(|f|(r)\right)}{\left[\exp\left\{(\log^{[q-1]}r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{\sigma_g^{(p,q)}(f)+\epsilon}}<\epsilon$$

$$i.e.,\qquad \log^{[p-1]}\left|\widehat{g}\right|\left(|f|(r)\right)<\log\epsilon+\left(\sigma_g^{(p,q)}(f)+\epsilon\right)\left\{\log^{[q-1]}r\right\}^{\rho_g^{(p,q)}(f)},$$

$$i.e.,\qquad \lim_{r\to\infty}\sup\frac{\log^{[p-1]}\left|\widehat{g}\right|\left(|f|(r)\right)}{(\log^{[q-1]}r)^{\rho_g^{(p,q)}(f)}}\le\sigma_g^{(p,q)}(f)+\epsilon.$$

Since $\epsilon>0$ is arbitrary, it follows that

$$\lim_{r\to\infty}\sup\frac{\log^{[p-1]}\left|\widehat{g}\right|\left(|f|(r)\right)}{(\log^{[q-1]}r)^{\rho_g^{(p,q)}(f)}}\le\sigma_g^{(p,q)}(f).\qquad(3.8)$$

On the other hand the divergence of the integral

$$\int_{r_0}^{\infty}\frac{\log^{[p-2]}\left|\widehat{g}\right|\left(|f|(r)\right)}{\left[\exp\left\{(\log^{[q-1]}r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{\sigma_g^{(p,q)}(f)-\epsilon+1}}dr,\quad(r_0>0)$$

implies that there exists a sequence of values of r tending to infinity such that

$$\frac{\log^{[p-2]}\left|\widehat{g}\right|\left(|f|(r)\right)}{\left[\exp\left\{(\log^{[q-1]}r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{\sigma_g^{(p,q)}(f)-\epsilon+1}}>\frac{1}{\left[\exp\left\{(\log^{[q-1]}r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{1+\epsilon}},$$

$$i.e.,\qquad \log^{[p-1]}\left|\widehat{g}\right|\left(|f|(r)\right)<\left(\sigma_g^{(p,q)}(f)-2\epsilon\right)\left\{\log^{[q-1]}r\right\}^{\rho_g^{(p,q)}(f)},$$

$$i.e.,\qquad \frac{\log^{[p-1]}\left|\widehat{g}\right|\left(|f|(r)\right)}{(\log^{[q-1]}r)^{\rho_g^{(p,q)}(f)}}>\left(\sigma_g^{(p,q)}(f)-2\epsilon\right).$$

Since $\epsilon>0$ is arbitrary, it follows from above that

$$\lim_{r\to\infty}\sup\frac{\log^{[p-1]}\left|\widehat{g}\right|\left(|f|(r)\right)}{(\log^{[q-1]}r)^{\rho_g^{(p,q)}(f)}}\ge\sigma_g^{(p,q)}(f).\qquad(3.9)$$

So from Equation (3.8) and Equation (3.9) we obtain that

$$\lim_{r\to\infty}\sup\frac{\log^{[p-1]}\left|\widehat{g}\right|\left(|f|(r)\right)}{(\log^{[q-1]}r)^{\rho_g^{(p,q)}(f)}}=\sigma_g^{(p,q)}(f).$$

This proves the theorem.

Remark 3.1 We give an example below which validates Theorem 3.3.

Example 1: Let $f(z)=z$, $g(z)=\log z$, $(z>0)$, $p=3$ and $q=2$. So $\hat{g}(z)=\exp(z)$

$$\rho_g^{(p,q)}(f)=\lim_{r\to\infty}\sup\frac{\log^{[p]}\left|\widehat{g}\right|\left(|f|(r)\right)}{(\log^{[q]}r)}=\lim_{r\to\infty}\sup\frac{\log^{[3]}\exp(r)}{\log^{[2]}r}$$

$$=\lim_{r\to\infty}\sup\frac{\log^{[2]}\exp(r)}{\log^{[2]}r}=1.$$

Again

$$\sigma_g^{(p,q)}(f) = \lim_{r\to\infty}\sup\frac{\log^{[p-1]}\widehat{|g|}(|f|(r))}{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}} = \lim_{r\to\infty}\sup\frac{\log^{[2]}\exp(r)}{\log^{[1]} r}$$
$$= \lim_{r\to\infty}\sup\frac{\log r}{\log r} = 1.$$

Next if we take $k = 2$, that is $k > \sigma_g^{(p,q)}(f)$ we see that the value of the integral for $r_0 > 0$,

$$\int_{r_0}^{\infty}\frac{\log^{[p-2]}\widehat{|g|}(|f|(r))}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{k+1}}dr = \int_{r_0}^{\infty}\frac{\log\exp(r)}{[\exp(\log r)]^{2+1}}dr = \int_{r_0}^{\infty}\frac{1}{r^2}dr = \left[\frac{1}{r}\right]_{r_0}^{\infty} = \frac{1}{r_0},$$

which is convergent. Next if we take $k = 0$, that is $k < \sigma_g^{(p,q)}(f)$ we see that the value of the integral for $r_0 > 0$,

$$\int_{r_0}^{\infty}\frac{\log^{[p-2]}\widehat{|g|}(|f|(r))}{\left[\exp\left\{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{k+1}}dr = \int_{r_0}^{\infty}\frac{\log\exp(r)}{[\exp(\log r)]^{0+1}}dr = \int_{r_0}^{\infty}\frac{r}{r}dr = [r]_{r_0}^{\infty} = \infty,$$

which is divergent.

Theorem 34.: Let $f, g \in$ A(K) be any two entire functions having finite positive (p,q)-th relative generalized lower order $\lambda_g^{(p,q)}(f)$, $\left(0 < \lambda_g^{(p,q)}(f) < \infty\right)$ and (p,q)-th relative generalized weak type $\tau_g^{(p,q)}(f)$ where p, q are any two positive integers. Then Definition 1.4 and Definition 1.5 are equivalent.

Proof: Case I: Let $\tau_g^{(p,q)}(f) = \infty$

Definition 1.4 ⟹ Definition 1.5.

As $\tau_g^{(p,q)}(f) = \infty$, from Definition (1.4) we get for an arbitrary positive G and for all sufficient large values of r that,

$$\log^{[p-1]}\widehat{|g|}(|f|(r)) > G.\left(\log^{[q-1]} r\right)^{\lambda_g^{(p,q)}(f)}$$
$$i.e., \quad \log^{[p-2]}\widehat{|g|}(|f|(r)) > \left[\exp\left\{\left(\log^{[q-1]} r\right)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{G}. \quad (3.10)$$

If possible, let the integral

$$\int_{r_0}^{\infty}\frac{\log^{[p-2]}\widehat{|g|}(|f|(r))}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{G+1}}dr, \quad (r_0 > 0)$$

be converge. Then by Lemma 2.1 we obtain that

$$\lim_{r\to\infty}\inf\frac{\log^{[p-2]}\widehat{|g|}(|f|(r))}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{G}} = 0.$$

So, for a sequence of values of r tending to infinity, we get that

$$\log^{[p-2]}\widehat{|g|}(|f|(r)) < \left[\exp\left\{\left(\log^{[q-1]} r\right)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{G}. \quad (3.11)$$

Therefore, by Equation (3.10) and Equation (3.11) we arrive at a contradiction.

Hence

$$\int_{r_0}^{\infty}\frac{\log^{[p-2]}\widehat{|g|}(|f|(r))}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{G+1}}dr, \quad (r_0 > 0)$$

is divergent, whenever G is finite which is Definition 1.5.
Now we show Definition 1.5 ⟹ Definition 1.4.
Let G be any positive number. Since $\tau_g^{(p,q)}(f)=\infty$, from Definition 1.5 the divergence of the integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]}\widehat{|g|}\big(|f|(r)\big)}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{G+1}} dr, \ \ (r_0>0)$$

gives an arbitrary positive ϵ and for all sufficiently large values of r that

$$\log^{[p-2]}\widehat{|g|}\big(|f|(r)\big) > \left[\exp\left\{\left(\log^{[q-1]} r\right)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{G-\epsilon}$$

$$\log^{[p-1]}\widehat{|g|}\big(|f|(r)\big) > (G-\epsilon)\left(\log^{[q-1]} r\right)^{\lambda_g^{(p,q)}(f)}$$

$$i.e., \qquad \frac{\log^{[p-1]}\widehat{|g|}\big(|f|(r)\big)}{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}} > G-\epsilon$$

$$i.e., \qquad \lim_{r\to\infty}\inf \frac{\log^{[p-1]}\widehat{|g|}\big(|f|(r)\big)}{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}} \geq G-\epsilon$$

Since $G>0$ is arbitrary, it follows that

$$\lim_{r\to\infty}\inf \frac{\log^{[p-1]}\widehat{|g|}\big(|f|(r)\big)}{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}} = \infty$$

$$i.e., \qquad \tau_g^{(p,q)}(f)=\infty$$

Thus Definition 1.4 follows.
Case II: Let $0 \leq \tau_g^{(p,q)}(f) < \infty$.
First we show that Definition 1.4 ⟹ Definition 1.5.
Sub case (A): $0 < \tau_g^{(p,q)}(f) < \infty$.
Let $f, g \in$ A(K) be any two entire functions such that

$$0 < \tau_g^{(p,q)}(f) < \infty$$

exists for positive integers p, q.
Then according to Definition 1.4 for any arbitrary positive ϵ and for large value of r we obtain that

$$\log^{[p-1]}\widehat{|g|}\big(|f|(r)\big) < \left(\tau_g^{(p,q)}(f)+\epsilon\right)\left\{\log^{[q-1]} r\right\}^{\lambda_g^{(p,q)}(f)}$$

$$i.e., \qquad \log^{[p-2]}\widehat{|g|}\big(|f|(r)\big) < \left[\exp\left\{\left(\log^{[q-1]} r\right)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{\left(\tau_g^{(p,q)}(f)+\epsilon\right)},$$

$$i.e., \qquad \frac{\log^{[p-2]}\widehat{|g|}\big(|f|(r)\big)}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{k}} < \frac{1}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{k-\left(\tau_g^{(p,q)}(f)+\epsilon\right)}}$$

Therefore,

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]}\widehat{|g|}\big(|f|(r)\big)}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{k+1}} dr, \ \ (r_0>0)$$

is convergent for $k > \tau_g^{(p,q)}(f)$.
Again, by Definition 1.4 we obtain for all sufficiently large values of r that

$$\log^{[p-1]}\widehat{|g|}\big(|f|(r)\big) > \left(\tau_g^{(p,q)}(f)-\epsilon\right)\left\{\log^{[q-1]} r\right\}^{\lambda_g^{(p,q)}(f)}$$

$$i.e., \quad \log^{[p-2]}\widehat{|g|}(|f|(r)) > \left[\exp\left\{\left(\log^{[q-1]} r\right)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{\left(\tau_g^{(p,q)}(f)-\epsilon\right)}, \tag{3.12}$$

so for $k < \tau_g^{(p,q)}(f)$, we get from Equation (3.12) that

$$\frac{\log^{[p-2]}\widehat{|g|}(|f|(r))}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{k}} > \frac{1}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{k-\left(\tau_g^{(p,q)}(f)-\epsilon\right)}}$$

Therefore,

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]}\widehat{|g|}(|f|(r))}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{k+1}} dr, \quad (r_0 > 0)$$

is divergent for $k < \tau_g^{(p,q)}(f)$.

Hence

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]}\widehat{|g|}(|f|(r))}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{k+1}} dr, \quad (r_0 > 0)$$

is convergent for $k > \tau_g^{(p,q)}(f)$ and divergent for $k < \tau_g^{(p,q)}(f)$.

Sub case (B): Let $\tau_g^{(p,q)}(f) = 0$.

When $\tau_g^{(p,q)}(f) = 0$ for positive integers p, q Definition (1.4) gives for a sequence of values of r tending to infinity that

$$\frac{\log^{[p-2]}\widehat{|g|}(|f|(r))}{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}} < \epsilon.$$

Then similar as before we get,

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]}\widehat{|g|}(|f|(r))}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{k+1}} dr, \quad (r_0 > 0)$$

is convergent for $k > 0$ and divergent for $k < 0$.

Thus, combining subcase (A) and subcase (B) Definition 1.5 follows.

Now we show Definition 1.5 $\Longrightarrow$ Definition 1.4.

From Definition 1.5 and arbitrary positive ϵ, the integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]}\widehat{|g|}(|f|(r))}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{\tau_g^{(p,q)}(f)+\epsilon+1}} dr, \quad (r_0 > 0)$$

is convergent. Then by Lemma 2.1 we get that

$$\liminf_{r\to\infty} \frac{\log^{[p-2]}\widehat{|g|}(|f|(r))}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{\tau_g^{(p,q)}(f)+\epsilon}} = 0.$$

So, we obtain for all sufficiently large values of r that

$$\frac{\log^{[p-2]}\widehat{|g|}(|f|(r))}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{\tau_g^{(p,q)}(f)+\epsilon}} < \epsilon$$

$$i.e.,\quad \log^{[p-2]}\widehat{|g|}\big(|f|(r)\big) < \epsilon.\left[\exp\left\{\left(\log^{[q-1]}r\right)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{\left(\tau_g^{(p,q)}(f)+\epsilon\right)}$$

$$i.e.,\quad \log^{[p-1]}\widehat{|g|}\big(|f|(r)\big) < \log\epsilon + \left(\tau_g^{(p,q)}(f)+\epsilon\right)\left\{\log^{[q-1]}r\right\}^{\lambda_g^{(p,q)}(f)}$$

$$i.e.,\quad \liminf_{r\to\infty}\frac{\log^{[p-1]}\widehat{|g|}\big(|f|(r)\big)}{(\log^{[q-1]}r)^{\lambda_g^{(p,q)}(f)}} \le \tau_g^{(p,q)}(f)+\epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}\widehat{|g|}\big(|f|(r)\big)}{(\log^{[q-1]}r)^{\lambda_g^{(p,q)}(f)}} \le \tau_g^{(p,q)}(f). \qquad (3.13)$$

On the other hand the divergence of the integral

$$\int_{r_0}^{\infty}\frac{\log^{[p-2]}\widehat{|g|}\big(|f|(r)\big)}{\left[\exp\left\{(\log^{[q-1]}r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{\tau_g^{(p,q)}(f)-\epsilon+1}}dr,\ \ (r_0 > 0)$$

implies for all sufficiently large values of r that

$$\frac{\log^{[p-2]}\widehat{|g|}\big(|f|(r)\big)}{\left[\exp\left\{(\log^{[q-1]}r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{\tau_g^{(p,q)}(f)-\epsilon+1}} > \frac{1}{\left[\exp\left\{(\log^{[q-1]}r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{1+\epsilon}}$$

$$i.e.,\quad \log^{[p-2]}\widehat{|g|}\big(|f|(r)\big) > \left[\exp\left\{\left(\log^{[q-1]}r\right)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{\tau_g^{(p,q)}(f)-2\epsilon}$$

$$i.e.,\quad \log^{[p-1]}\widehat{|g|}\big(|f|(r)\big) < \left(\tau_g^{(p,q)}(f)-2\epsilon\right)\left\{\log^{[q-1]}r\right\}^{\lambda_g^{(p,q)}(f)}$$

$$i.e.,\quad \frac{\log^{[p-1]}\widehat{|g|}\big(|f|(r)\big)}{(\log^{[q-1]}r)^{\lambda_g^{(p,q)}(f)}} > \left(\tau_g^{(p,q)}(f)-2\epsilon\right).$$

Since $\epsilon > 0$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}\widehat{|g|}\big(|f|(r)\big)}{(\log^{[q-1]}r)^{\lambda_g^{(p,q)}(f)}} \ge \tau_g^{(p,q)}(f). \qquad (3.14)$$

So from Equation (3.13) and Equation (3.14) we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}\widehat{|g|}\big(|f|(r)\big)}{(\log^{[q-1]}r)^{\lambda_g^{(p,q)}(f)}} = \tau_g^{(p,q)}(f).$$

This proves the theorem.

Remark 3.2 We give an example below which validates Theorem 3.4.

Example 2: Let $f(z) = z$, $g(z) = \log z\,,(z > 0)$, $p = 3$ and $q = 2$. So $\hat{g}(z) = \exp(z)$

$$\lambda_g^{(p,q)}(f) = \liminf_{r\to\infty}\frac{\log^{[p]}\widehat{|g|}\big(|f|(r)\big)}{(\log^{[q]}r)} = \liminf_{r\to\infty}\frac{\log^{[3]}\exp(r)}{\log^{[2]}r} = \liminf_{r\to\infty}\frac{\log^{[2]}r}{\log^{[2]}r} = 1.$$

Again

$$\tau_g^{(p,q)}(f) = \liminf_{r\to\infty}\frac{\log^{[p-1]}\widehat{|g|}\big(|f|(r)\big)}{(\log^{[q-1]}r)^{\lambda_g^{(p,q)}(f)}} = \liminf_{r\to\infty}\frac{\log^{[2]}\exp(r)}{\log^{[1]}r} = \liminf_{r\to\infty}\frac{\log r}{\log r} = 1.$$

Next if we take $k = 2$, that is $k > \tau_g^{(p,q)}(f)$ we see that the value of the integral for $r_0 > 0$,

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} \widehat{|g|}(|f|(r))}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{k+1}} dr = \int_{r_0}^{\infty} \frac{\log \exp(r)}{[\exp(\log r)]^{2+1}} dr = \int_{r_0}^{\infty} \frac{r}{r^3} dr = \left[\frac{1}{r}\right]_{r_0}^{\infty} = \frac{1}{r_0},$$

which is convergent. Next if we take $k = 0$, that is $k < \tau_g^{(p,q)}(f)$ we see that the value of the integral for $r_0 > 0$,

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} \widehat{|g|}(|f|(r))}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{k+1}} dr = \int_{r_0}^{\infty} \frac{\log \exp(r)}{[\exp(\log r)]^{0+1}} dr = \int_{r_0}^{\infty} \frac{r}{r} dr = [r]_{r_0}^{\infty} = \infty,$$

which is divergent.

Theorem 3.5: Let $f, g \in \mathrm{A(K)}$ be any two entire functions having finite positive (p,q)-th relative generalized order $\rho_g^{(p,q)}(f)$, $\left(0 < \rho_g^{(p,q)}(f) < \infty\right)$ and (p,q)-th relative generalized lower type $\bar{\sigma}_g^{(p,q)}(f)$ where p, q are any positive integers. Then Definition 2.1 and Definition 2.2 are equivalent.

Proof: With the help of Lemma 2.1 and similar to the proof of Theorem 3.1 we can prove the above theorem.

Theorem 3.6: Let $f, g \in \mathrm{A(K)}$ be any two entire functions having finite positive (p,q)-th relative generalized lower order $\lambda_g^{(p,q)}(f)$, $\left(0 < \lambda_g^{(p,q)}(f) < \infty\right)$ and the growth indicator $\bar{\tau}_g^{(p,q)}(f)$ where p, q are any two positive integers. Then Definition 2.3 and Definition 2.4 are equivalent.

Proof: With the help of Lemma 2.1 and similar to the proof of Theorem 3.2 we can prove the above theorem.

Theorem 3.7: Let $f, g \in \mathrm{A(K)}$ be any two entire functions such that f is regular (p,q)-th relative generalized growth with respect to g, i.e.,

$$\rho_g^{(p,q)}(f) = \lambda_g^{(p,q)}(f), \qquad \left(0 < \lambda_g^{(p,q)}(f) = \rho_g^{(p,q)}(f) < \infty\right),$$

where p, q are any two positive integers. Then the following quantities

(i) $\sigma_g^{(p,q)}(f)$, (ii) $\tau_g^{(p,q)}(f)$, (iii) $\bar{\sigma}_g^{(p,q)}(f)$ (iv) $\bar{\tau}_g^{(p,q)}(f)$

are all equivalent.

Proof: From Definition 1.5 it follows that the integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} \widehat{|g|}(|f|(r))}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{k+1}} dr, \quad (r_0 > 0)$$

is convergent for $k > \tau_g^{(p,q)}(f)$ and divergent for $k < \tau_g^{(p,q)}(f)$.

On the other hand Definition (1.3) implies that the integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} \widehat{|g|}(|f|(r))}{\left[\exp\left\{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{k+1}} dr, \quad (r_0 > 0)$$

is convergent for $k > \sigma_g^{(p,q)}(f)$ and divergent for $k < \sigma_g^{(p,q)}(f)$.

We show (i)$\Longrightarrow$(ii)

Now it is obvious that all the quantities in the expression

$$\left[\frac{\log^{[p-2]} \widehat{|g|}(|f|(r))}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{k+1}} - \frac{\log^{[p-2]} \widehat{|g|}(|f|(r))}{\left[\exp\left\{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{k+1}}\right]$$

are non-negative type. So,

$$\int_{r_0}^{\infty}\left[\frac{\log^{[p-2]}\widehat{[g]}\left(|f|(r)\right)}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{k+1}}-\frac{\log^{[p-2]}\widehat{[g]}\left(|f|(r)\right)}{\left[\exp\left\{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{k+1}}\right]dr \geq 0, \ \ (r_0>0)$$

$$i.e., \quad \int_{r_0}^{\infty}\frac{\log^{[p-2]}\widehat{[g]}\left(|f|(r)\right)}{\left[\exp\left\{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}\right\}\right]^{k+1}}dr \geq \int_{r_0}^{\infty}\frac{\log^{[p-2]}\widehat{[g]}\left(|f|(r)\right)}{\left[\exp\left\{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}\right\}\right]^{k+1}}dr, \ \ (r_0>0)$$

$$i.e., \quad \tau_g^{(p,q)}(f) \geq \sigma_g^{(p,q)}(f). \tag{3.15}$$

Further, f is of regular (p,q)-th relative generalized growth with respect to g. So, we get

$$\sigma_g^{(p,q)}(f)=\lim_{r\to\infty}\sup\frac{\log^{[p-1]}\widehat{[g]}\left(|f|(r)\right)}{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}} \geq \lim_{r\to\infty}\inf\frac{\log^{[p-1]}\widehat{[g]}\left(|f|(r)\right)}{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}}=\lim_{r\to\infty}\inf\frac{\log^{[p-1]}\widehat{[g]}\left(|f|(r)\right)}{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}}$$
$$=\tau_g^{(p,q)}(f)$$

$$i.e., \quad \sigma_g^{(p,q)}(f) \geq \tau_g^{(p,q)}(f). \tag{3.16}$$

Combining Equation (3.15) and Equation (3.16) we get

$$\sigma_g^{(p,q)}(f)=\tau_g^{(p,q)}(f). \tag{3.17}$$

Now we show (ii)⟹(iii)

Since f is of regular (p,q)-th relative generalized growth with respect to g. So, we get

$$\tau_g^{(p,q)}(f)=\lim_{r\to\infty}\inf\frac{\log^{[p-1]}\widehat{[g]}\left(|f|(r)\right)}{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}}=\lim_{r\to\infty}\inf\frac{\log^{[p-1]}\widehat{[g]}\left(|f|(r)\right)}{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}}=\bar{\sigma}_g^{(p,q)}(f)$$

$$i.e., \quad \tau_g^{(p,q)}(f)=\bar{\sigma}_g^{(p,q)}(f). \tag{3.18}$$

Now we show (ii)⟹(iii)

From Equation (3.17), Equation (3.18) and the condition

$$\rho_g^{(p,q)}(f)=\lambda_g^{(p,q)}(f)$$

it follows that

$$i.e., \quad \bar{\sigma}_g^{(p,q)}(f)=\sigma_g^{(p,q)}(f). \tag{3.19}$$

So,

$$\bar{\sigma}_g^{(p,q)}(f)=\lim_{r\to\infty}\sup\frac{\log^{[p-1]}\widehat{[g]}\left(|f|(r)\right)}{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}}=\lim_{r\to\infty}\sup\frac{\log^{[p-1]}\widehat{[g]}\left(|f|(r)\right)}{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}}=\bar{\tau}_g^{(p,q)}(f)$$

$$i.e., \quad \bar{\sigma}_g^{(p,q)}(f)=\bar{\tau}_g^{(p,q)}(f).$$

Now we show (iv)⟹(i)

Since f is of regular (p,q)-th relative generalized growth with respect to g, we get that

$$\bar{\tau}_g^{(p,q)}(f)=\lim_{r\to\infty}\sup\frac{\log^{[p-1]}\widehat{[g]}\left(|f|(r)\right)}{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}}=\lim_{r\to\infty}\sup\frac{\log^{[p-1]}\widehat{[g]}\left(|f|(r)\right)}{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}}=\sigma_g^{(p,q)}(f)$$

$$i.e., \quad \bar{\tau}_g^{(p,q)}(f)=\sigma_g^{(p,q)}(f). \tag{3.20}$$

Thus the theorem follows.

4. Conclusion

In the line of the works as carried out in the paper one may think of finding out integral representation of relative $(p,q,t)L^{th}\Psi$ growth and $(p,q,t)L^{*}\Psi$ growth of entire and meromorphic function with respect to another one and this treatment can be done under the flavour of bicomplex analysis. As a consequence, the derivation of relevant results is still open to the future workers of this branch.

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